

Common Fixed Points of Weakly Commuting Multivalued Mappings on Domain of Sets Endowed with Directed Graph

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Abstract: In this paper, the existence of coincidence points and common fixed points for multivalued mappings satisfying certain graphic ψ -contraction contractive conditions with set-valued domain endowed with a graph, without appealing to continuity, is established. Some examples are presented to support the results proved herein. Our results unify, generalize and extend various results in the existing literature.

Keywords and Phrases: multivalued mapping, domain of sets, coincidence point, common fixed point, graph ψ -contraction pair, directed graph.

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1 Introduction and preliminaries

Order oriented fixed point theory is studied in an environment created by a class of partially ordered sets with appropriate mappings satisfying certain order condition like monotonicity, expansivity or order continuity. Existence of fixed points in partially ordered metric spaces has been studied by Ran and Reurings [26]. Recently, many researchers have obtained fixed point results for single and multivalued mappings defined on partially ordered metrics spaces (see, e.g., [6, 8, 18, 24]). Jachymski and Jozwik [19] introduced a new approach in metric fixed point theory by replacing the order structure with a graph structure on a metric space. In this way, the results proved in ordered

metric spaces are generalized (see also [20] and the reference therein); in fact, in 2010, Gwóźdz-lukawska and Jachymski [17], developed the Hutchinson-Barnsley theory for finite families of mappings on a metric space endowed with a directed graph. Abbas and Nazir [2] obtained some fixed point results for power graph contraction pair endowed with a graph. Bojor [13] proved fixed point theorems for Reich type contractions on metric spaces with a graph. For more results in this direction, we refer to [4, 5, 12, 14, 15, 25] and reference mentioned therein.

Beg and Butt [9] proved the existence of fixed points of multivalued mapping in metric spaces endowed with a graph G . Recently, Abbas et al., [1] obtained fixed points of set valued mappings satisfying certain graphic contraction conditions with set valued domain endowed with a graph. Nicolae et al. [25] established some fixed points of multivalued generalized contractions in metric spaces endowed with a graph.

The aim of this paper is to prove some coincidence point and common fixed point results for discontinuous multivalued graphic ψ -contractive mappings defined on the family of closed and bounded subsets of a metric space endowed with a graph G . These results extend and strengthen various comparable results in the existing literature [1, 9, 12, 19, 20, 23] .

Consistent with Jachymski [20], let (X, d) be a metric space and Δ denotes the diagonal of $X \times X$. Let G be a directed graph, such that the set $V(G)$ of its vertices coincides with X and $E(G)$ be the set of edges of the graph which contains all loops, that is, $\Delta \subseteq E(G)$. Also assume that the graph G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$.

Definition 1.1. [20] An operator $f : X \rightarrow X$ is called a Banach G -contraction or simply G -contraction if

- (a) f preserves edges of G ; for each $x, y \in X$ with $(x, y) \in E(G)$, we have $(f(x), f(y)) \in E(G)$,
- (b) f decreases weights of edges of G ; there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$, we have $d(f(x), f(y)) \leq \alpha d(x, y)$.

If x and y are vertices of G , then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}$ ($n \in \{0, 1, 2, \dots, k\}$) of vertices such that $x_0 = x$, $x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1, 2, \dots, k\}$.

Notice that a graph G is connected if there is a directed path between any two vertices and it is weakly connected if \tilde{G} is connected, where \tilde{G} denotes the

undirected graph obtained from G by ignoring the direction of edges. Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

It is more convenient to treat \tilde{G} as a directed graph for which the set of its edges is symmetric, under this convention; we have that

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

In $V(G)$, we define the relation R in the following way:

For $x, y \in V(G)$, we have xRy if and only if, there is a path in G from x to y . If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, the equivalence class $[x]_{\tilde{G}}$ in $V(G)$ defined by the relation R is $V(G_x)$.

Recall that if $f : X \rightarrow X$ is an operator, then by F_f we denote the set of all fixed points of f . Set

$$X_f := \{x \in X : (x, f(x)) \in E(G)\}.$$

Jachymski [19] used the following property:

(P) : for any sequence $\{x_n\}$ in X , if $x_n \rightarrow x$ as $n \rightarrow \infty$ and $(x_n, x_{n+1}) \in E(G)$, then $(x_n, x) \in E(G)$.

Theorem 1.2. [19] Let (X, d) be a complete metric space and G a directed graph such that $V(G) = X$ and $f : X \rightarrow X$ a G -contraction. Suppose that $E(G)$ and the triplet (X, d, G) have property (P). Then the following statements hold:

- (i) $F_f \neq \emptyset$ if and only if $X_f \neq \emptyset$;
- (ii) if $X_f \neq \emptyset$ and G is weakly connected, then f is a Picard operator, i.e., $F_f = \{x^*\}$ and sequence $\{f^n(x)\} \rightarrow x^*$ as $n \rightarrow \infty$, for all $x \in X$;
- (iii) for any $x \in X_f$, $f|_{[x]_{\tilde{G}}}$ is a Picard operator;
- (iv) if $X_f \subseteq E(G)$, then f is a weakly Picard operator, i.e., $F_f \neq \emptyset$ and, for each $x \in X$, we have sequence $\{f^n(x)\} \rightarrow x^* \in F_f$ as $n \rightarrow \infty$.

For detailed discussion on Picard operators, we refer to Berinde ([10, 11]).

Let (X, d) be a metric space and $CB(X)$ a class of all nonempty closed and bounded subsets of X . For $A, B \in CB(X)$, let

$$H(A, B) = \max\{\sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B)\},$$

where $d(x, B) = \inf\{d(x, b) : b \in B\}$ is the distance of a point x to the set B . The mapping H is said to be the Pompeiu-Hausdorff metric induced by d .

Throughout this paper, we assume that a directed graph G has no parallel edge and G is a weighted graph in the sense that each vertex x is assigned the weight $d(x, x) = 0$ and each edge (x, y) is assigned the weight $d(x, y)$. Since d is a metric on X , the weight assigned to each vertex x to vertex y need not be zero and, whenever a zero weight is assigned to some edge (x, y) , it reduces to a loop (x, x) having weight 0. Further, in Pompeiu-Hausdorff metric induced by metric d , the Pompeiu-Hausdorff weight assigned to each $U, V \in CB(X)$ need not be zero (that is, $H(U, V) \neq 0$) and, whenever a zero Pompeiu-Hausdorff weight is assigned to some $U, V \in CB(X)$, then it reduces to $U = V$.

Definition 1.3. [1] Let A and B be two nonempty subsets of X . Then by:

- (a) ‘there is an edge between A and B ’, we mean there is an edge between some $a \in A$ and $b \in B$ which we denote by $(A, B) \subset E(G)$.
- (b) ‘there is a path between A and B ’, we mean that there is a path between some $a \in A$ and $b \in B$.

In $CB(X)$, we define a relation R in the following way:

For $A, B \in CB(X)$, we have ARB if and only if, there is a path between A and B .

We say that the relation R on $CB(X)$ is transitive if there is a path between A and B , and there is a path between B and C , then there is a path between A and C .

Consider the mapping $T : CB(X) \rightarrow CB(X)$ instead of a mapping T from X to X or from X to $CB(X)$.

For mappings $T : CB(X) \rightarrow CB(X)$, the set X_T is defined as

$$X_T := \{U \in CB(X) : (U, T(U)) \subseteq E(G)\}.$$

Recently, Abbas et al. [1] gave the following definition.

Definition 1.4. Let $T : CB(X) \rightarrow CB(X)$ a be multivalued mapping. The mapping T is said to be a graph ϕ -contraction if the following conditions hold:

- (i) There is an edge between A and B implies there is an edge between $T(A)$ and $T(B)$ for all $A, B \in CB(X)$.
- (ii) There is a path between A and B implies there is a path between $T(A)$ and $T(B)$ for all $A, B \in CB(X)$.
- (iii) There exists an upper semi-continuous and nondecreasing function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\phi(t) < t$ for each $t > 0$ such that there is an edge between A and B implies that

$$H(T(A), T(B)) \leq \phi(H(A, B)) \text{ for all } A, B \in CB(X). \quad (1.1)$$

Definition 1.5. Let $S, T : CB(X) \rightarrow CB(X)$ be two multivalued mappings. The set $U \in CB(X)$ is said to be a coincidence point of S and T , if $S(U) = T(U)$. Also, a set $A \in CB(X)$ is said to be a fixed point of S if $S(A) = A$. The set of all coincidence points of S and T is denoted by $CP(S, T)$ and the set of all fixed points of S is denoted by $Fix(S)$.

Definition 1.6. Two maps $S, T : CB(X) \rightarrow CB(X)$ are said to be weakly compatible if the commute at their coincidence point.

For more details to the weakly compatible maps, we refer the reader to [3, 21, 22].

A subset Γ of $CB(X)$ is said to be complete if for any set $X, Y \in \Gamma$, there is an edge between X and Y .

Abbas et al. [1] used the property P^* stated as follows: A graph G is said to have property

- (P^*) : if for any sequence $\{X_n\}$ in $CB(X)$ with $X_n \rightarrow X$ as $n \rightarrow \infty$, there exists edge between X_{n+1} and X_n for $n \in \mathbb{N}$, implies that there is a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ with an edge between X and X_{n_k} for $n \in \mathbb{N}$.

Theorem 1.6. [1] Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. If $T : CB(X) \rightarrow CB(X)$ is a graph ϕ -contraction mapping such that the relation R on $CB(X)$ is transitive, then following statements hold:

- (a) if $Fix(T)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $U, V \in Fix(T)$ is 0.
- (b) $X_T \neq \emptyset$ provided that $Fix(T) \neq \emptyset$.
- (c) If $X_T \neq \emptyset$ and the weakly connected graph G satisfies the property (P^*) , then T has a fixed point.
- (d) $Fix(T)$ is complete if and only if $Fix(T)$ is a singleton.

In the sequel, the letters \mathbb{R} , \mathbb{R}^+ and \mathbb{N} denote the set of all real numbers, the set of all positive real numbers and the set of all natural numbers, respectively.

We denote Ψ the set of all functions $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, where ψ is nondecreasing function with $\sum_{i=1}^{\infty} \psi^n(t)$ is convergent. It is easy to show that if $\psi \in \Psi$, then $\psi(t) < t$ for any $t > 0$.

We now give the following definition:

Definition 1.7. Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$, $E(G) \supseteq \Delta$ and for every U in $CB(X)$, $(S(U), U) \subseteq E(G)$ and $(U, T(U)) \subseteq E(G)$. Let $S, T : CB(X) \rightarrow CB(X)$ be two multivalued mappings. The pair (S, T) of maps is said to be

- (I) graph ψ_1 -contraction pair if there exists a $\psi \in \Psi$, there is an edge between A and B such that

$$H(S(A), S(B)) \leq \psi(M_1(A, B)) \text{ holds,}$$

where

$$M_1(A, B) = \max\{H(T(A), T(B)), H(S(A), T(A)), H(S(B), T(B)), \frac{H(S(A), T(B)) + H(S(B), T(A))}{2}\}.$$

- (II) graph ψ_2 -contraction pair if there exists a $\psi \in \Psi$, there is an edge between A and B such that

$$H(S(A), S(B)) \leq \psi(M_2(A, B)) \text{ holds,}$$

where

$$M_2(A, B) = \alpha H(T(A), T(B)) + \beta H(S(A), T(A)) + \gamma H(S(B), T(B)) + \delta_1 H(S(A), T(B)) + \delta_2 H(S(B), T(A))$$

and $\alpha, \beta, \gamma, \delta_1, \delta_2 \geq 0$, $\delta_1 \leq \delta_2$ with $\alpha + \beta + \gamma + \delta_1 + \delta_2 \leq 1$.

It is obvious that if a pair (S, T) of multivalued mappings on $CB(X)$ is a graph ψ_1 -contraction or graph ψ_2 -contraction for graph G , then pair (S, T) is also graph ψ_1 -contraction or graph ψ_2 -contraction respectively, for the graphs G^{-1} , \tilde{G} and G_0 , here the graph G_0 is defined by $E(G_0) = X \times X$.

Definition 1.8. A metric space (X, d) is called an ε -chainable metric space for some $\varepsilon > 0$ if for given $x, y \in X$, there is $n \in \mathbb{N}$ and a sequence $\{x_n\}$ such that

$$x_0 = x, x_n = y \text{ and } d(x_{i-1}, x_i) < \varepsilon \text{ for } i = 1, \dots, n.$$

For fixed point result of mappings defined on ε -chainable metric space, we refer to [9] and references mentioned therein.

We also need of the following lemma of Nadler [23] (see also, [7]).

Lemma 1.9. Let (X, d) be a metric space. If $U, V \in CB(X)$ with $H(U, V) < \varepsilon$, then for each $u \in U$ there exists an element $v \in V$ such that $d(u, v) < \varepsilon$.

2 Common Fixed Points

In this section, we obtain coincidence point and common fixed point results for multivalued selfmaps on $CB(X)$ satisfying graph ψ -contraction conditions endow with a directed graph.

Theorem 2.1. Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$, $E(G) \supseteq \Delta$ and $S, T : CB(X) \rightarrow CB(X)$ a graph ψ_1 -contraction pair such that the range of T contains the range of S . Then the following statements hold:

- (i) $CP(S, T) \neq \emptyset$ provided that G is weakly connected with satisfies the property (P^*) and $T(X)$ is complete subspace of $CB(X)$.
- (ii) if $CP(S, T)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $S(U)$ and $S(V)$ is 0 for all $U, V \in CP(S, T)$.
- (iii) if $CP(S, T)$ is complete and S and T are weakly compatible, then $Fix(S) \cap Fix(T)$ is a singleton.
- (iv) $Fix(S) \cap Fix(T)$ is complete if and only if $Fix(S) \cap Fix(T)$ is a singleton.

Proof. To prove (i), let A_0 be an arbitrary element in $CB(X)$. Since range of T contains the range of S , chosen $A_1 \in CB(X)$ such that $S(A_0) = T(A_1)$. Continuing this process, having chosen A_n in $CB(X)$, we obtain an A_{n+1} in $CB(X)$ such that $S(x_n) = T(x_{n+1})$ for $n \in \mathbb{N}$. The inclusion $(A_{n+1}, T(A_{n+1})) \subseteq E(G)$ and $(T(A_{n+1}), A_n) = (S(A_n), A_n) \subseteq E(G)$ implies that $(A_{n+1}, A_n) \subseteq E(G)$.

We may assume that $S(A_n) \neq S(A_{n+1})$ for all $n \in \mathbb{N}$. If not, then $S(A_{2k}) = S(A_{2k+1})$ for some k , implies $T(A_{2k+1}) = S(A_{2k+1})$, and thus $A_{2k+1} \in CP(S, T)$. Now, since $(A_{n+1}, A_n) \subseteq E(G)$ for all $n \in \mathbb{N}$, and pair (S, T) form a graph ψ_1 -contraction, so we have

$$\begin{aligned} H(T(A_{n+1}), T(A_{n+2})) &= H(S(A_n), S(A_{n+1})) \\ &\leq \psi(M_1(A_n, A_{n+1})), \end{aligned}$$

where

$$\begin{aligned} &M_1(A_n, A_{n+1}) \\ &= \max\left\{H(T(A_n), T(A_{n+1})), H(S(A_n), T(A_n)), H(S(A_{n+1}), T(A_{n+1})), \right. \\ &\quad \left. \frac{H(S(A_n), T(A_{n+1})) + H(S(A_{n+1}), T(A_n))}{2}\right\} \\ &= \max\left\{H(T(A_n), T(A_{n+1})), H(T(A_{n+1}), T(A_n)), H(T(A_{n+2}), T(A_{n+1})), \right. \\ &\quad \left. \frac{H(T(A_{n+1}), T(A_{n+1})) + H(T(A_{n+2}), T(A_n))}{2}\right\} \\ &\leq \max\left\{H(T(A_n), T(A_{n+1})), H(T(A_{n+1}), T(A_{n+2})), \right. \\ &\quad \left. \frac{H(T(A_{n+2}), T(A_{n+1})) + H(T(A_{n+1}), T(A_n))}{2}\right\} \\ &= \max\{H(T(A_n), T(A_{n+1})), H(T(A_{n+1}), T(A_{n+2}))\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} H(T(A_{n+1}), T(A_{n+2})) &\leq \psi(\max\{H(T(A_n), T(A_{n+1})), H(T(A_{n+1}), T(A_{n+2}))\}) \\ &= \psi(H(T(A_n), T(A_{n+1}))) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore for $i = 1, 2, \dots, n$, we have

$$\begin{aligned} H(T(A_{i-1}), T(A_i)) &\leq \psi(H(A_{i-1}, A_i)), \\ H(T(A_{i-2}), T(A_{i-1})) &\leq \psi(H(A_{i-2}, A_{i-1})), \\ &\dots, \\ H(T(A_0), T(A_1)) &\leq \psi(H(A_0, A_1)), \end{aligned}$$

and so we obtain

$$H(T(A_n), T(A_{n+1})) \leq \psi^n(H(A_0, T(A_1)))$$

for all $n \in \mathbb{N}$. Now for $m, n \in \mathbb{N}$ with $m > n \geq 1$, we have

$$\begin{aligned} H(T(A_n), T(A_m)) &\leq H(T(A_n), T(A_{n+1})) + H(T(A_{n+1}), T(A_{n+2})) \\ &\quad + \dots + H(T(A_{m-1}), T(A_m)) \\ &\leq \psi^n(H(A_0, T(A_1))) + \psi^{n+1}(H(A_0, T(A_1))) \\ &\quad + \dots + \psi^{m-1}(H(A_0, T(A_1))). \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \psi^i(H(A_0, T(A_1)))$, we get $H(T(A_n), T(A_m)) \rightarrow 0$ as $n, m \rightarrow \infty$. Therefore $\{T(A_n)\}$ is a Cauchy sequence in $T(X)$. Since $(T(X), d)$ is complete in $CB(X)$, we have $T(A_n) \rightarrow V$ as $n \rightarrow \infty$ for some $V \in CB(X)$. Also, we can find U in $CB(X)$ such that $T(U) = V$.

We claim that $S(U) = T(U)$. If not, then since $(T(A_{n+1}), T(A_n)) \subseteq E(G)$ so by property (P*), there exists a subsequence $\{T(A_{n_k+1})\}$ of $\{T(A_{n+1})\}$ such that $(T(U), T(A_{n_k+1})) \subseteq E(G)$ for every $n \in \mathbb{N}$. As $(U, T(U)) \subseteq E(G)$ and $(T(A_{n_k+1}), A_{n_k}) = (S(A_{n_k}), A_{n_k}) \subseteq E(G)$ implies that $(U, A_{n_k}) \subseteq E(G)$. Now

$$H(S(U), T(A_{n_k+1})) = H(S(U), S(A_{n_k})) \leq \psi(M_1(U, A_{n_k})), \quad ((1))$$

where

$$\begin{aligned} M_1(U, A_{n_k}) &= \max\left\{H(T(U), T(A_{n_k})), H(S(U), T(U)), H(S(A_{n_k}), T(A_{n_k})), \right. \\ &\quad \left. \frac{H(S(U), T(A_{n_k})) + H(S(A_{n_k}), T(U))}{2}\right\} \\ &= \max\left\{H(T(U), T(A_{n_k})), H(S(U), T(U)), H(T(A_{n_k+1}), T(A_{n_k})), \right. \\ &\quad \left. \frac{H(S(U), T(A_{n_k})) + H(T(A_{n_k+1}), T(U))}{2}\right\}. \end{aligned}$$

Now we consider the following cases:

If $M_1(U, A_{n_k}) = H(T(U), T(A_{n_k}))$, then on taking limit as $k \rightarrow \infty$ in (1), we have

$$H(S(U), T(U)) \leq \psi(H(T(U), T(U))),$$

a contradiction.

When $M_1(U, A_{n_k}) = H(S(U), T(U))$, then

$$H(S(U), T(U)) \leq \psi(H(S(U), T(U))),$$

gives a contradiction.

In case $M_1(U, A_{n_k}) = H(T(A_{n_k+1}), T(A_{n_k}))$, then on taking limit as $k \rightarrow \infty$ in (1), we get

$$H(S(U), T(U)) \leq \psi(H(T(U), T(U))),$$

a contradiction.

Finally, if $M_1(U, A_{n_k}) = \frac{H(S(U), T(A_{n_k})) + H(T(A_{n_k+1}), T(U))}{2}$, then on taking limit as $k \rightarrow \infty$, we have

$$\begin{aligned} H(S(U), T(U)) &\leq \psi\left(\frac{H(S(U), T(U)) + H(T(U), T(U))}{2}\right) \\ &= \psi\left(\frac{H(S(U), T(U))}{2}\right), \end{aligned}$$

a contradiction.

Hence $S(U) = T(U)$, that is, $U \in CP(S, T)$.

To prove (ii), suppose that $CP(S, T)$ is complete set in G . Let $U, V \in CP(S, T)$ and suppose that the Pompeiu-Hausdorff weight assign to the $S(U)$ and $S(V)$ is not zero. Since pair (S, T) is a graph ψ_1 -contraction, we obtain that

$$\begin{aligned} &H(S(U), S(V)) \\ &\leq \psi(M_1(U, V)) \\ &\leq \psi(\max\{H(T(U), T(V)), H(S(U), T(U)), H(S(V), T(V)), \\ &\quad \frac{H(S(U), T(V)) + H(S(V), T(U))}{2}\}) \\ &= \psi(\max\{H(S(U), S(V)), H(S(U), S(U)), H(S(V), T(V)), \\ &\quad \frac{H(S(U), S(V)) + H(S(V), S(U))}{2}\}) \\ &= \psi(H(S(U), S(V))), \end{aligned}$$

a contradiction as $\psi(t) < t$ for all $t > 0$. Hence (ii) is proved.

To prove (iii), suppose the set $CP(S, T)$ is weakly compatible. First we are to show that $Fix(T) \cap Fix(S)$ is nonempty. Let $W = S(U) = T(U)$, then we have $T(W) = TS(U) = ST(U) = S(W)$, which shows that $W \in CP(S, T)$. Thus the Pompeiu-Hausdorff weight assign to the $S(U)$ and $S(W)$ is zero (by ii). Hence $W = S(W) = T(W)$, that is, $W \in$

$Fix(S) \cap Fix(T)$. Since $CP(S, T)$ is singleton set, implies $Fix(S) \cap Fix(T)$ is singleton.

Finally to prove (iv), suppose the set $Fix(S) \cap Fix(T)$ is complete. We are to show that $Fix(T) \cap Fix(S)$ is singleton. Assume on contrary that there exist $U, V \in CB(X)$ such that $U, V \in Fix(S) \cap Fix(T)$ and $U \neq V$. By completeness of $Fix(S) \cap Fix(T)$, there exists an edge between U and V . As pair (S, T) is a graph ψ_1 -contraction, so we have

$$\begin{aligned}
H(U, V) &= H(S(U), S(V)) \\
&\leq \psi(M_1(U, V)) \\
&= \psi(\max\{H(T(U), T(V)), H(S(U), T(U)), H(S(V), T(V)), \\
&\quad \frac{H(S(U), T(V)) + H(S(V), T(U))}{2}\}) \\
&= \psi(\max\{H(U, V), H(U, U), H(V, V), \\
&\quad \frac{H(U, V) + H(V, U)}{2}\}) \\
&= \psi(H(U, V)),
\end{aligned}$$

a contradiction. Hence $U = V$. Conversely, if $Fix(S) \cap Fix(T)$ is singleton, then since $E(G) \supseteq \Delta$, so it is obvious that $F(S) \cap F(T)$ is complete set. \square

Example 2.2. Let $X = \{1, 2, \dots, n\} = V(G)$, $n > 2$ and $E(G) = \{(i, j) \in X \times X : i \leq j\}$. Let $V(G)$ be endowed with metric $d : X \times X \rightarrow \mathbb{R}^+$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{1}{n} & \text{if } x \in \{1, 2\} \text{ with } x \neq y, \\ \frac{n}{n+1} & \text{otherwise.} \end{cases}$$

Furthermore, the Pompeiu-Hausdorff metric is given by

$$H(A, B) = \begin{cases} \frac{1}{n} & \text{if } A, B \subseteq \{1, 2\} \text{ with } A \neq B, \\ \frac{n}{n+1} & \text{if } A \text{ or } B \text{ (or both)} \not\subseteq \{1, 2\} \text{ with } A \neq B, \\ 0 & \text{if } A = B. \end{cases}$$

The Pompeiu-Hausdorff weights (for $n = 4$) assigned to $A, B \in CB(X)$ are shown in the Figure.

Define $S, T : CB(X) \rightarrow CB(X)$ as follows:

$$\begin{aligned} S(U) &= \begin{cases} \{1\}, & \text{if } U \subseteq \{1, 2\}, \\ \{1, 2\}, & \text{if } U \not\subseteq \{1, 2\} \end{cases} \\ T(U) &= \begin{cases} \{1\}, & \text{if } U = \{1\}, \\ \{1, 2, 3\}, & \text{if } U \subseteq \{2, 3\}. \\ \{1, 2, \dots, n\}, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that, for all $V \in CB(X)$, $(V, S(V)) \subseteq E(G)$ and $(V, T(V)) \subseteq E(G)$. Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by

$$\psi(\alpha) = \begin{cases} \frac{1}{2}\alpha^2 & 0 \leq \alpha < \frac{1}{2} \\ \frac{\alpha}{\alpha + 1}, & \frac{1}{2} \leq \alpha. \end{cases}$$

It is easy to verify that $\psi \in \Psi$. Now for all $A, B \in CB(X)$ with $S(A) \neq S(B)$, we consider the following cases:

(i) If $A \subseteq \{1, 2\}$ and $B = \{3\}$ with $(A, B) \subseteq E(G)$, then we have

$$\begin{aligned}
H(S(A), S(B)) &= H(\{1\}, \{1, 2\}) \\
&= \frac{1}{n} \\
&< \frac{n}{2n+1} \\
&= \psi\left(\frac{n}{n+1}\right) \\
&= \psi(H(\{1, 2\}, \{1, 2, 3\})) \\
&= \psi(H(S(B), T(B))) \leq \psi(M_1(A, B)).
\end{aligned}$$

(ii) When $A \subseteq \{1, 2\}$ and $B \subsetneq \{1, 2, 3\}$ with $(A, B) \subseteq E(G)$, implies that

$$\begin{aligned}
H(S(A), S(B)) &= H(\{1\}, \{1, 2\}) \\
&= \frac{1}{n} \\
&< \frac{n}{2n+1} \\
&= \psi\left(\frac{n}{n+1}\right) \\
&= \psi(H(\{1, 2\}, \{1, 2, \dots, n\})) \\
&= \psi(H(S(B), T(B))) \leq \psi(M_1(A, B)).
\end{aligned}$$

(iii) In case $A = \{3\}$ and $B \subseteq \{1, 2\}$ and with $(A, B) \subseteq E(G)$, we have

$$\begin{aligned}
H(S(A), S(B)) &= H(\{1, 2\}, \{1\}) \\
&= \frac{1}{n} \\
&< \frac{n}{2n+1} \\
&= \psi\left(\frac{n}{n+1}\right) \\
&= \psi(H(\{1, 2\}, \{1, 2, 3\})) \\
&= \psi(H(S(A), T(A))) \leq \psi(M_1(A, B)).
\end{aligned}$$

(iv) When $A \subsetneq \{1, 2, 3\}$ and $B \subseteq \{1, 2\}$ with $(A, B) \subseteq E(G)$, implies that

$$\begin{aligned}
H(S(A), S(B)) &= H(\{1, 2\}, \{1\}) \\
&= \frac{1}{n} \\
&< \frac{n}{2n+1} \\
&= \psi\left(\frac{n}{n+1}\right) \\
&= \psi(H(\{1, 2\}, \{1, 2, \dots, n\})) \\
&= \psi(H(S(A), T(A))) \leq \psi(M_1(A, B)).
\end{aligned}$$

Hence pair (S, T) is graph ψ_1 -contraction. Thus all the conditions of Theorem 1 are satisfied. Moreover, $\{1\}$ is the common fixed point of S and T , and $Fix(S) \cap Fix(T)$ is complete. \square

In the next example we show that it is not necessary the given graph $(V(G), E(G))$ will always be complete graph.

Example 2.3. Let $X = \{1, 2, \dots, n\} = V(G)$, $n > 2$ and

$$\begin{aligned}
E(G) &= \{(1, 1), (2, 2), \dots, (n, n), \\
&\quad (1, 2), \dots, (1, n)\}.
\end{aligned}$$

On $V(G)$, the metric $d : X \times X \rightarrow \mathbb{R}^+$ and Pompeiu-Hausdorff metric $H : CB(X) \rightarrow \mathbb{R}^+$ are defined as in Example 2.2. The Pompeiu-Hausdorff

weights (for $n = 4$) assigned to $A, B \in CB(X)$ are shown in the Figure.

Define $S, T : CB(X) \rightarrow CB(X)$ as follows:

$$\begin{aligned} S(U) &= \begin{cases} \{1\}, & \text{if } U = \{1\}, \\ \{1, 2\}, & \text{if } U \neq \{1\} \end{cases} \\ T(U) &= \begin{cases} \{1\}, & \text{if } U = \{1\}, \\ \{1, \dots, n\}, & \text{if } U \neq \{1\}. \end{cases} \end{aligned}$$

Note that, $(S(A), A) \subseteq E(G)$ and $(A, T(A)) \subseteq E(G)$ for all $A \in CB(X)$.

Take $\psi(\alpha) = \begin{cases} \frac{1}{8}t, & t \in [0, \frac{1}{4}] \\ \frac{t+1}{t+2}, & t \geq \frac{1}{4}. \end{cases}$. Note that $\psi \in \Psi$.

For all $A, B \in CB(X)$ with $S(A) \neq S(B)$, we consider the following cases:

(I) If $A = \{1\}$ and $B \neq \{1\}$, then we have

$$\begin{aligned}
H(S(A), S(B)) &= \frac{1}{n} \\
&< \frac{2n+1}{3n+1} \\
&= \psi\left(\frac{n}{n+1}\right) \\
&= \psi(H(S(B), T(B))) \leq \psi(M_1(A, B)).
\end{aligned}$$

(II) If $A \neq \{1\}$ and $B = \{1\}$, then we have

$$\begin{aligned}
H(S(A), S(B)) &= \frac{1}{n} \\
&< \frac{2n+1}{3n+1} \\
&= \psi\left(\frac{n}{n+1}\right) \\
&= \psi(H(S(A), T(A))) \leq \psi(M_1(A, B)).
\end{aligned}$$

Hence pair (S, T) is graph ψ_1 -contraction. Thus all the conditions of Theorem 1 are satisfied. Moreover, S and T have a common fixed point and $Fix(S) \cap Fix(T)$ is complete in $CB(X)$. \square

Theorem 2.4. Let (X, d) be a ε -chainable complete metric space for some $\varepsilon > 0$ and $S, T : CB(X) \rightarrow CB(X)$ be multivalued mappings. Suppose that for all $A, B \in CB(X)$,

$$0 < H(S(A), S(B)) < \varepsilon$$

and there exists a $\psi \in \Psi$ such

$$H(S(A), S(B)) \leq \psi(M_1(A, B)),$$

hold where

$$\begin{aligned}
M_1(A, B) &= \max\{H(T(A), T(B)), H(S(A), T(A)), H(S(B), T(B)), \\
&\quad \frac{H(S(A), T(B)) + H(S(B), T(A))}{2}\}.
\end{aligned}$$

Then S and T have a common fixed point provided that S and T are weakly compatible.

Proof. By Lemma 1.9, from $H(A, B) < \varepsilon$, we have for each $a \in A$, an element $b \in B$ such that $d(a, b) < \varepsilon$. Consider the graph G as $V(G) = X$ and

$$E(G) = \{(a, b) \in X \times X : 0 < d(a, b) < \varepsilon\}.$$

Then the ε -chainability of (X, d) implies that G is connected. For $(A, B) \subset E(G)$, we have from the hypothesis

$$H(S(A), S(B)) \leq \psi(M_1(A, B)),$$

$$\text{where } M_1(A, B) = \max\left\{H(S(A), T(B)), H(S(A), T(A)), H(S(B), T(B)), \frac{H(S(A), T(B)) + H(S(B), T(A))}{2}\right\}$$

implies that pair (S, T) is graph ψ_1 -contraction.

Also, G has property (P^*) . Indeed, if $\{X_n\}$ in $CB(X)$ with $X_n \rightarrow X$ as $n \rightarrow \infty$ and $(X_n, X_{n+1}) \subset E(G)$ for $n \in \mathbb{N}$, implies that there is a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that $(X_{n_k}, X) \subset E(G)$ for $n \in \mathbb{N}$. So by Theorem 2.1 (iii), S and T have a common fixed point. \square

Corollary 2.5. Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. Suppose that the mapping $S : CB(X) \rightarrow CB(X)$ satisfies the following:

- (a) for every V in $CB(X)$, $(V, S(V)) \subset E(G)$.
- (b) There exists $\psi \in \Psi$ such that there is an edge between A and B implies that

$$H(S(A), S(B)) \leq \psi(M_1(A, B)),$$

where

$$M_1(A, B) = \max\left\{H(A, B), H(A, S(A)), H(B, S(B)), \frac{H(A, S(B)), H(B, S(A))}{2}\right\}.$$

Then following statements hold:

- (i) if $Fix(S)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $U, V \in Fix(S)$ is 0.

- (ii) If the weakly connected graph G satisfies the property (P^*) , then S has a fixed point.
- (iii) $Fix(S)$ is complete if and only if $Fix(S)$ is a singleton.

Proof. Take $T = I$ (identity map) in (1.2), then Corollary 2.5 follows from Theorem 2.1. \square

Theorem 2.6. Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$, $E(G) \supseteq \Delta$ and $S, T : CB(X) \rightarrow CB(X)$ a graph ψ_2 -contraction pair such that the range of T contains the range of S . Then the following statements hold:

- (i) $CP(S, T) \neq \emptyset$ provided that G is weakly connected with satisfies the property (P^*) and $T(X)$ is complete subspace of $CB(X)$.
- (ii) if $CP(S, T)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $S(U)$ and $S(V)$ is 0 for all $U, V \in CP(S, T)$.
- (iii) if $CP(S, T)$ is complete and S and T are weakly compatible, then $Fix(S) \cap Fix(T)$ is a singleton.
- (iv) $Fix(S) \cap Fix(T)$ is complete if and only if $Fix(S) \cap Fix(T)$ is a singleton.

Proof. To prove (i), let A_0 be an arbitrary element in $CB(X)$. Since range of T contains the range of S , chosen $A_1 \in CB(X)$ such that $S(A_0) = T(A_1)$. Continuing this process, having chosen A_n in $CB(X)$, we obtain an A_{n+1} in $CB(X)$ such that $S(x_n) = T(x_{n+1})$ for $n \in \mathbb{N}$. The inclusion $(A_{n+1}, T(A_{n+1})) \subseteq E(G)$ and $(T(A_{n+1}), A_n) = (S(A_n), A_n) \subseteq E(G)$ implies that $(A_{n+1}, A_n) \subseteq E(G)$.

We may assume that $S(A_n) \neq S(A_{n+1})$ for all $n \in \mathbb{N}$. If not, then $S(A_{2k}) = S(A_{2k+1})$ for some k , implies $T(A_{2k+1}) = S(A_{2k+1})$, and thus $A_{2k+1} \in CP(S, T)$. Now, since $(A_{n+1}, A_n) \subseteq E(G)$ for all $n \in \mathbb{N}$, and pair (S, T) form a graph ψ_2 -contraction, so we have

$$\begin{aligned} H(T(A_{n+1}), T(A_{n+2})) &= H(S(A_n), S(A_{n+1})) \\ &\leq \psi(M_2(A_n, A_{n+1})), \end{aligned}$$

where

$$\begin{aligned}
& M_2(A_n, A_{n+1}) \\
&= \alpha H(T(A_n), T(A_{n+1})) + \beta H(S(A_n), T(A_n)) + \gamma H(S(A_{n+1}), T(A_{n+1})) \\
&\quad \delta_1 H(S(A_n), T(A_{n+1})) + \delta_2 H(S(A_{n+1}), T(A_n)) \\
&= \alpha H(T(A_n), T(A_{n+1})) + \beta H(T(A_{n+1}), T(A_n)) + \gamma H(T(A_{n+2}), T(A_{n+1})) \\
&\quad \delta_1 H(T(A_{n+1}), T(A_{n+1})) + \delta_2 H(T(A_{n+2}), T(A_n)) \\
&\leq (\alpha + \beta) H(T(A_n), T(A_{n+1})) + \gamma H(T(A_{n+1}), T(A_{n+2})), \\
&\quad \delta_2 [H(T(A_{n+2}), T(A_{n+1})) + H(T(A_{n+1}), T(A_n))] \\
&= (\alpha + \beta + \delta_2) H(T(A_n), T(A_{n+1})) + (\gamma + \delta_2) H(T(A_{n+1}), T(A_{n+2})).
\end{aligned}$$

Now, if $H(T(A_n), T(A_{n+1})) \leq H(T(A_{n+1}), T(A_{n+2}))$, we have

$$\begin{aligned}
H(T(A_{n+1}), T(A_{n+2})) &\leq \psi(\max\{H(T(A_n), T(A_{n+1})), H(T(A_{n+1}), T(A_{n+2}))\}) \\
&= \psi(H(T(A_n), T(A_{n+1})))
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore for $i = 1, 2, \dots, n$, we have

$$\begin{aligned}
H(T(A_{i-1}), T(A_i)) &\leq \psi(H(A_{i-1}, A_i)), \\
H(T(A_{i-2}), T(A_{i-1})) &\leq \psi(H(A_{i-2}, A_{i-1})), \\
&\dots, \\
H(T(A_0), T(A_1)) &\leq \psi(H(A_0, A_1)),
\end{aligned}$$

and so we obtain

$$H(T(A_n), T(A_{n+1})) \leq \psi^n(H(A_0, T(A_1)))$$

for all $n \in \mathbb{N}$. Follows the similar argument to those in the proof of Theorem 2.1, we get $H(T(A_n), T(A_m)) \rightarrow 0$ as $n, m \rightarrow \infty$. Therefore $\{T(A_n)\}$ is a Cauchy sequence in $T(X)$. Since $(T(X), d)$ is complete in $CB(X)$, we have $T(A_n) \rightarrow V$ as $n \rightarrow \infty$ for some $V \in CB(X)$. Also, we can find U in $CB(X)$ such that $T(U) = V$.

We claim that $S(U) = T(U)$. If not, then since $(T(A_{n+1}), T(A_n)) \subseteq E(G)$ so by property (P*), there exists a subsequence $\{T(A_{n_k+1})\}$ of $\{T(A_{n+1})\}$ such that $(T(U), T(A_{n_k+1})) \subseteq E(G)$ for every $n \in \mathbb{N}$. As $(U, T(U)) \subseteq E(G)$ and $(T(A_{n_k+1}), A_{n_k}) = (S(A_{n_k}), A_{n_k}) \subseteq E(G)$ implies that $(U, A_{n_k}) \subseteq E(G)$. Now

$$\begin{aligned}
H(S(U), T(A_{n_k+1})) &= H(S(U), S(A_{n_k})) \\
&\leq \psi(M_2(U, A_{n_k})),
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
M_2(U, A_{n_k}) &= \alpha H(T(U), T(A_{n_k})) + \beta H(S(U), T(U)) + \gamma H(S(A_{n_k}), T(A_{n_k})) \\
&\quad + \delta_1 H(S(U), T(A_{n_k})) + \delta_2 H(S(A_{n_k}), T(U)) \\
&= \alpha H(T(U), T(A_{n_k})) + \beta H(S(U) + T(U)) + \gamma H(T(A_{n_k+1}), T(A_{n_k})) \\
&\quad + \delta_1 H(S(U), T(A_{n_k})) + \delta_2 H(T(A_{n_k+1}), T(U)).
\end{aligned}$$

On taking limit as $k \rightarrow \infty$ in (2.2), we have

$$\begin{aligned}
H(S(U), T(U)) &\leq \psi((\beta + \delta_1) H(T(U), T(U))) \\
&< H(S(U), T(U)),
\end{aligned}$$

a contradiction. Hence $S(U) = T(U)$, that is, $U \in CP(S, T)$.

To prove (ii), suppose that $CP(S, T)$ is complete set in G . Let $U, V \in CP(S, T)$ and suppose that the Pompeiu-Hausdorff weight assign to the $S(U)$ and $S(V)$ is not zero. Since pair (S, T) is a graph ψ_2 -contraction, we obtain that

$$H(S(U), S(V)) \leq \psi(M_2(U, V)), \quad (2.3)$$

where

$$\begin{aligned}
M_2(U, V) &= \alpha H(T(U), T(V)) + \beta H(S(U), T(U)) + \gamma H(S(V), T(V)) \\
&\quad + \delta_1 H(S(U), T(V)) + \delta_2 H(S(V), T(U)) \\
&= \alpha H(S(U), S(V)) + \beta H(S(U), S(U)) + \gamma H(S(V), T(V)) \\
&= (\alpha + \delta_1 + \delta_2) H(S(U), S(V)),
\end{aligned}$$

thus

$$\begin{aligned}
H(S(U), S(V)) &\leq \psi((\alpha + \delta_1 + \delta_2) H(S(U), S(V))) \\
&< \psi(H(S(U), S(V))),
\end{aligned}$$

a contradiction as $\psi(t) < t$ for all $t > 0$. Hence (ii) is proved.

To prove (iii), suppose the set $CP(S, T)$ is weakly compatible. First we are to show that $Fix(T) \cap Fix(S)$ is nonempty. Let $W = S(U) = T(U)$, then we have $T(W) = TS(U) = ST(U) = S(W)$, which shows that $W \in CP(S, T)$. Thus the Pompeiu-Hausdorff weight assign to the $S(U)$ and $S(W)$ is zero (by ii). Hence $W = S(W) = T(W)$, that is, $W \in Fix(S) \cap Fix(T)$. Since $CP(S, T)$ is singleton set, implies $Fix(S) \cap Fix(T)$ is singleton.

Finally to prove (iv), suppose the set $Fix(S) \cap Fix(T)$ is complete. We are to show that $Fix(T) \cap Fix(S)$ is singleton. Assume on contrary that there exist $U, V \in CB(X)$ such that $U, V \in Fix(S) \cap Fix(T)$ and $U \neq V$. By completeness of $Fix(S) \cap Fix(T)$, there exists an edge between U and V . As pair (S, T) is a graph ψ_2 -contraction, so we have

$$\begin{aligned}
H(U, V) &= H(S(U), S(V)) \\
&\leq \psi(M_2(U, V)) \\
&= \psi(\alpha H(T(U), T(V)) + \beta H(S(U), T(U)) + \gamma H(S(V), T(V)) \\
&\quad + \delta_1 H(S(U), T(V)) + \delta_2 H(S(V), T(U))) \\
&= \psi(\alpha H(U, V) + \beta H(U, U) + \gamma H(V, V) + \delta_1 H(U, V) + \delta_2 H(V, U)) \\
&\leq \psi(H(U, V)),
\end{aligned}$$

a contradiction. Hence $U = V$. Conversely, if $Fix(S) \cap Fix(T)$ is singleton, then since $E(G) \supseteq \Delta$, so it is obvious that $F(S) \cap F(T)$ is complete set. \square

Example 2.7. Let $X = \mathbb{R}_+ = V(G)$ be endowed with Euclidean metric d . Let $f : X \rightarrow X$ be defined as $f(x) = \begin{cases} 10, & \text{if } x \in [0, 10] \\ 20, & \text{otherwise} \end{cases}$ and $(a, b) \in E(G)$ for some $a \in A, b \in B$ if $b = f(a)$. Define $S, T : CB(X) \rightarrow CB(X)$ as follows:

$$\begin{aligned}
S(U) &= \begin{cases} [0, 10], & \text{if } U \subseteq [0, 10] \\ [10, 20], & \text{otherwise} \end{cases} \quad \text{and} \\
T(U) &= \begin{cases} [0, 10], & \text{if } U \subseteq [0, 10] \\ [5, 25], & \text{otherwise.} \end{cases}
\end{aligned}$$

Note that, for all $V \in CB(X)$, $(S(V), V) \subseteq E(G)$ and $(V, T(V)) \subseteq E(G)$. Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by

$$\psi(\alpha) = \begin{cases} \frac{3}{4}t & 0 \leq t < 1 \\ \frac{5}{6}t, & 1 \leq t. \end{cases}$$

It is easy to verify that $\psi \in \Psi$. Now for all $A, B \in CB(X)$ with $S(A) \neq$

$S(B)$, we consider $A \subseteq [0, 10]$ and $B \not\subseteq [0, 10]$ with $(A, B) \subseteq E(G)$, implies

$$\begin{aligned}
H(S(A), S(B)) &= H([0, 10], [10, 20]) \\
&= 10 \\
&< \frac{100}{9} \\
&= \psi(15\alpha + 5\beta) \\
&= \psi(\alpha H([0, 10], [5, 25]) + \gamma H([10, 20], [5, 25])) \\
&= \psi(\alpha H(T(A), T(B)) + \gamma H(S(B), T(B))) \leq \psi(M_2(A, B)),
\end{aligned}$$

where $\alpha = \frac{5}{6}, \gamma = \frac{1}{6}, \beta = \delta_1 = \delta_2 = 0$ and

$$\begin{aligned}
M_2(A, B) &= \alpha H(T(A), T(B)) + \beta H(S(A), T(A)) + \gamma H(S(B), T(B)) \\
&\quad + \delta_1 H(S(A), T(B)) + \delta_2 H(S(B), T(A))
\end{aligned}$$

Hence pair (S, T) is graph ψ_2 -contraction. Thus all the conditions of Theorem 2.6 are satisfied. Moreover, the set $[0, 10]$ is the common fixed point of S and T , and $Fix(S) \cap Fix(T)$ is complete. \square

The following corollary generalizes and extends Theorem 2.1 of [1].

Corollary 2.7. Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. Suppose that the mappings $S, T : CB(X) \rightarrow CB(X)$ satisfies the following:

- (a) for every V in $CB(X)$, $(S(V), V) \subset E(G)$ and $(V, T(V)) \subseteq E(G)$.
- (b) There exists $\psi \in \Psi$ such that for all $A, B \in CB(X)$ with there is an edge between A and B implies

$$H(S(A), S(B)) \leq \psi(\alpha H(T(A), T(B)) + \beta H(S(A), T(A)) + \gamma H(S(B), T(B)))$$

hold, where α, β, γ are nonnegative real numbers with $\alpha + \beta + \gamma \leq 1$. If the range of T contains the range of S , then the following statements hold:

- (i) $CP(S, T) \neq \emptyset$ provided that G is weakly connected with satisfies the property (P^*) and $T(X)$ is complete subspace of $CB(X)$.
- (ii) if $CP(S, T)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $S(U)$ and $S(V)$ is 0 for all $U, V \in CP(S, T)$.

- (iii) if $CP(S, T)$ is complete and S and T are weakly compatible, then $Fix(S) \cap Fix(T)$ is a singleton.
- (iv) $Fix(S) \cap Fix(T)$ is complete if and only if $Fix(S) \cap Fix(T)$ is a singleton.

Corollary 2.8. Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. Suppose that the mappings $S : CB(X) \rightarrow CB(X)$ satisfies the following:

- (a) for every V in $CB(X)$, $(S(V), V) \in E(G)$.
- (b) There exists $\psi \in \Psi$ such that for all $A, B \in CB(X)$ with there is an edge between A and B implies

$$H(S(A), S(B)) \leq \psi(\alpha H(A, B) + \beta H(S(A), A) + \gamma H(B, S(B)))$$

hold, where α, β, γ are nonnegative real numbers with $\alpha + \beta + \gamma \leq 1$. Then the following statements hold:

- (i) if $Fix(S)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $U, V \in Fix(S)$ is 0.
- (ii) If the weakly connected graph G satisfies the property (P^*) , then S has a fixed point.
- (iii) $Fix(S)$ is complete if and only if $Fix(S)$ is a singleton.

Proof. If we take $T = I$ (identity map) in above Corollary 2, the result follows.

Remark 2.9.

- (1) If $E(G) := X \times X$, then clearly G is connected and our Theorem 2.1 improves and generalizes Theorem 2.1 in [1], Theorem 2.1 in [9], Theorem 3.1 in [19].
- (2) If $E(G) := X \times X$, then clearly G is connected and our Theorem 2.4 extends and generalizes Theorem 2.5 in [9], Theorem 3.2 in [23], Theorem 5.1 in [16] and Theorem 3.1 in [19].

- (3) If $E(G) := X \times X$, then clearly G is connected and our Corollary 2.5 improves and generalizes Theorem 2.1 in [9], Theorem 3.2 in [23] and Theorem 3.1 in [19].

Conclusion. Jachymski and Jozwik initiated the study of ordered structured metric fixed point theory by using the ordered structured with a graph structure on a metric space. Recently many results appeared in the literature giving the fixed point problems of mappings endowed with a graph. We presented the common fixed points of a class of multivalued maps with set-valued domain that are commuting only at their coincidence points endowed with a directed graph. We presented some examples to show the validated of obtained results.

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